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CHARACTERISTIC NUMBERS FOR UNORIENTED Z-HOMOLOGY MANIFOLDS

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ABSTRACT. It is shown that the analogue of Thom's theorem on Stiefel-Whitney numbers holds for $\mathbf{Z}$-homology manifolds

INTRODUCTION

Thom's theorem on Stiefel-Whitney numbers says that a smooth manifold is determined, up to unoriented cobordism, by its Stiefel-Whitney numbers. We are interested here in the analogue for $\mathbf{Z}$-homology manifolds. Such manifolds also possess characteristic numbers, as follows.

Martin and Maunder [M-M] have defined the tangent bundle of a $\mathbf{Z}$-homology n-manifold M by taking a regular neighbourhood of the diagonal in $M \times M$ and making it into a "homology cobordism D^n-bundle". This gives rise to a classifying map $M \to BH(n)$, where $BH(n)$ is the classifying space for such bundles. Stabilizing, one obtains the stable tangent bundle of M, classified by a map $t_M\colon M \to BH$. The image $(t_M)_*[M]$ of the fundamental class $[M]$ lies in $H_n(BH; \mathbf{Z}/2)$, and is, by definition, the "characteristic numbers of M". It is an invariant of cobordism. The main result of this paper is that the analogue of Thom's theorem holds for $\mathbf{Z}$-homology manifolds.

Theorem. *Let M be a $\mathbf{Z}$-homology n-manifold without boundary (M need not be orientable). Let $t_M\colon M \to BH$ be the classifying map of the stable tangent bundle in the sense of* [M-M] *and* [Maunder]. *Then $(t_M)_*[M] = 0$ in $H_n(BH, \mathbf{Z}/2)$ if and only if M is the boundary of a $\mathbf{Z}$-homology manifold.*

The proof we give here is an adaptation of the argument of [B-H]. An essential ingredient of this proof is the use of the projective bundle construction, and this makes it necessary to work with bundles with involution, stably at least. For the equivariant bundle theory one has classifying spaces $B\overline{H}(n)$ and $B\overline{H}$. "Forgetting involutions" induces a map $F\colon B\overline{H} \to BH$, and this map has a section $S\colon BH \to B\overline{H}$, corresponding to the fact that, stably at least, there is

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a natural way to impose free involutions on homology cobordism bundles. The stable tangent bundle of M may be provided with an involution T by taking a regular neighborhood of the diagonal $\times\{0\}$ in $M \times M \times [-1, 1]^k$ invariant with respect to the involution $T(x, y, v) = (y, x, -v)$. For big k this gives rise to a classifying map $\bar{t}_M: M \to B\overline{H}$. One may now apply the methods of [B-H]. Namely, given that $(\bar{t}_M)_*[M] = 0$, one shows M is a boundary. However, all one needs, in fact, is that $(t_M)_*[M] = 0$. For, following [Mann and Miller], one shows that $\bar{t}_M$ and $S \circ t_M$ are homotopic. Since S is a section of $F: B\overline{H} \to BH$, S_* is a monomorphism and the theorem follows.

In the last section we indicate an extension to characteristic numbers of homology manifolds with a fixed system of singularities.

1. Homology cobordism Z/2-bundles

In this section we adapt the construction, given in [M-M], of the classifying space BH, for homology cobordism bundles, to the equivariant case, obtaining a classifying space $B\overline{H}$. As far as possible we stick to the notation of [M-M] and [Maunder].

First we extend the notion of homology cobordism bundle to the case of $\mathbf{Z}/2$-bundles.

An *involution*, on a PL space X, is a PL map $f: X \to X$ such that $f \circ f$ is the identity. For example, let $D^k = [-1, 1]^k$ with involution $v \to -v$. The restriction of this to the boundary S^{k-1} is a free involution.

A space E over K which satisfies (i) but not necessarily (ii) of Definition 3.1 of [M-M] will be referred to as a *prebundle*, meaning, roughly speaking, that it is a "not necessarily locally trivial homology cobordism S^n-bundle".

An *involution* on a prebundle is an involution on the total space which maps each block to itself.

A $\mathbf{Z}/2$-*prebundle* is a prebundle together with an involution as above. Two such are *isomorphic* if there exists an isomorphism G between them, in the sense of [M-M], together with a free involution on the total space of G which maps each block of G to itself and which restricts to the two given involutions.

The *product* $\mathbf{Z}/2$ S^n-*bundle*, ε, is defined by $\varepsilon(C) = C \times S^n$ with involution $(x, v) \Leftrightarrow (x, -v)$, E is *trivial* if it is isomorphic (as a $\mathbf{Z}/2$-prebundle) to a product ε. E is *locally trivial* if, for each cell C of the base, the restriction of E to the cell-complex consisting of C and all its subcells is trivial.

Finally, a $\mathbf{Z}/2$ S^n-bundle is a locally trivial $\mathbf{Z}/2$-prebundle. Two such are *isomorphic* if they are isomorphic as $\mathbf{Z}/2$-prebundles.

$\mathbf{Z}/2$ D^n-bundles are defined similarly. The involution on E is required to be free on the *associated sphere bundle* $\mathrm{Sph}(E)$. Sometimes we shall use the terminology "equivariant bundle" instead of $\mathbf{Z}/2$-bundle.

The results of [M-M] carry over to the $\mathbf{Z}/2$ case as follows.

First it is straightforward to verify that Theorem 4.11 of [M-M] carries over to the $\mathbf{Z}/2$ case. One has a classifying Δ-set, $B\overline{H}(n)$, for the Δ-semigroup,

$\overline{H}(n)$, whose k-simplexes are the automorphisms of the product $\Delta^k \times S^{n-1}$. We denote by $B\overline{H}(n)$ also the realization, $|B\overline{H}(n)|$, of $B\overline{H}(n)$. The $\mathbf{Z}/2$ version of 4.11 for simplicial complexes then reads as follows.

Theorem 4.11. *Let K be a simplicial complex with underlying space $|K|$. Then the set, $A\overline{H}_n(K)$, of isomorphism classes of $\mathbf{Z}/2$ S^{n-1}-bundles (or, equivalently, D^n-bundles) over K is in natural* 1-1 *correspondence with homotopy classes of maps of $|K|$ into $B\overline{H}(n)$.*

As in [M-M] Theorem 4.11 is proved using only a particular case of Theorem 4.5 which carries over to the $\mathbf{Z}/2$ case. On the other hand, it seems likely that Theorem 4.5 is only valid stably in the $\mathbf{Z}/2$ case (see below).

In [M-M] oriented bundles are also used. The classifying space, $BSH(n)$, for these is simply connected, a fact which is essential for the application of obstruction theory at various points. There does not seem to be a $\mathbf{Z}/2$ analogue of this but we can get around the difficulty by stabilizing as follows.

Let E be a $\mathbf{Z}/2$ D^n-bundle. The $\mathbf{Z}/2$ D^{n+1}-bundle, $E \times D^1$, is defined by $E \times D^1(C) = E(C) \times D^1$. Similarly one defines $E \times D^k$. It is the $\times$ product of E with the product D^k-bundle over a point. This notation should not be confused with the notation $(E|X) \times I$ of [M-M].

"*Stable*" will mean "stable with respect to $\times D^k$". We write $K_{\overline{H}}(X)$ for the set of *stable isomorphism classes* of $\mathbf{Z}/2$-bundles over X.

For stable $\mathbf{Z}/2$ disk-bundles over a simplicial complex K we define *Whitney sum* as follows. The definition 2.1 of Whitney sum and Theorem 2.2 of [Maunder] carry over to the $\mathbf{Z}/2$ case when the base K is a simplicial complex or, more generally, a Δ-set. It is enough to observe that the cells of $K \times K$ are of the form (simplex of K) $\times$ (simplex of K) and that therefore Q (on p. 104 of [M-M]) is cellularly collapsible. It follows that the bundle D (ibid.) is trivial.

Whitney sum is compatible with stabilization:

Lemma. *Let E and F be $\mathbf{Z}/2$ disk-bundles over a simplicial complex. Then*

$$(E \oplus F) \times D^1 \cong (E \times D^1) \oplus F.$$

Proof. Recall from [Maunder] that $E \oplus F = \Delta^* G$ where $\mathrm{Am}(G) = E \times F$ (Am being "amalgamation"). Since Am clearly commutes with Δ^* and $\times D^1$ one has

$$(E \oplus F) \times D^1 = (\Delta^* G) \times D^1 \cong \Delta^*(G \times D^1).$$

Also

$$\mathrm{Am}(G \times D^1) \cong \mathrm{Am}(G) \times D^1 \cong (E \times F) \times D^1 \cong (E \times D^1) \times F.$$

Hence the lemma.

Corollary. $E \times D^k \cong E \oplus \varepsilon^k$.

Proof. As in [Maunder], $E \oplus \varepsilon^0 \cong E$. Thus

$$E \times D^k \cong (E \oplus \varepsilon^0) \times D^k \cong E \oplus (\varepsilon^0 \times D^k) = E \oplus \varepsilon^k.$$

Thus we have an induced Whitney sum, $\oplus$, on $K_{\overline{H}}(K)$, for K a simplicial complex. Now, as in [Maunder], there exists an H-space $B\overline{H}$ and a natural isomorphism of abelian groups, between $K_{\overline{H}}(K)$ and homotopy classes of maps of $|K|$ into $B\overline{H}$.

One can now prove a $\mathbf{Z}/2$ version of Theorem 4.5 of [M-M]. Suppose K is a homology cell complex and L a subcomplex all of whose cells are simplexes.

Theorem 4.5 (Stable). *Given a $\mathbf{Z}/2$ disk-bundle E over K, there exists, for big k, a $\mathbf{Z}/2$ disk-bundle F over K' (the simplicial complex underlying K) such that $F|L = E \times D^k|L$ and* $\mathrm{Am}(F) \cong E \times D^k$ *by an isomorphism whose restriction to L is $(E|L) \times [0, 1]$.*

Proof. The proof follows closely the proof of Theorem 4.5 of [M-M], duly modified to allow for involutions. The only point to note is that, as on p. 107 of [M-M], Q collapses onto a codimension-one subcomplex X, which is acyclic. Since $\dim X = \dim Q - 1$, we may assume, by induction on the dimension of the base, K, that for big r, $D \times D^r|X \cong \mathrm{Am}(F)$ for some $\mathbf{Z}/2$-bundle F over K'.

Since $B\overline{H}$ is an H-space and $\widetilde{H}_*(X, \mathbf{Z}) = 0$, any map of X into $B\overline{H}$ is nulhomotopic by standard obstruction theory (for $\pi_1(B\overline{H})$ acts trivially on $\pi_i(B\overline{H})$, for $i \geq 1$, and, by the universal coefficient theorem, $\widetilde{H}_*(X, G) = 0$ for any coefficient G). Thus any $\mathbf{Z}/2$ bundle over X is stably trivial so that, for big s, $F \times D^s$ is trivial which, in turn, implies that, for big k, $D \times D^k|X$ is trivial. Now the proof proceeds as in [M-M].

Theorem 4.5 (Stable) implies the following.

Theorem 4.12 (Stable). *Amalgamation* $\mathrm{Am}\colon K_{\overline{H}}(X') \to K_{\overline{H}}(X)$ *is a bijection for X a homology cell complex and X' its underlying simplicial complex.*

Thus $K_{\overline{H}}(X)$ is in 1-1 correspondence with homotopy classes of maps of X into $B\overline{H}$, and the H-space structure on $B\overline{H}$ corresponds to Whitney sum on $K_{\overline{H}}(X)$. This contrasts with the fact (see above) that Whitney sum is also defined unstably when X is a simplicial complex.

2. $\mathbf{Z}/2$ NORMAL BUNDLES

Let M be a $\mathbf{Z}$-homology manifold properly embedded in a $\mathbf{Z}$-homology manifold Q, and suppose that M is the fixed-point set of an involution on Q. We will show that, stably, the involution is a bundle involution near M.

To do this we triangulate Q so that the involution f is simplicial and M is a full subcomplex of Q. Write Q and M for the underlying simplicial complexes. Take dual complexes so that, for each simplex σ of M, we have $D(\sigma, M) \subset D(\sigma, Q)$ and, if $\sigma \in \partial M$, then also $D(\sigma, \partial M) \subset D(\sigma, \partial Q)$. Define a $\mathbf{Z}/2$ disk-prebundle $\underline{E}(M, Q)$ (or E for short) over the dual homology cell complex M^* by $E(D(\sigma, M)) = D(\sigma, Q)$ and $E(D(\sigma, \partial M)) = D(\sigma, \partial Q)$ so that the total space of E (resp. $E/\partial M^*$) is the *simplicial* neighborhood

$\mathrm{N}(M', Q')$ (resp. $\mathrm{N}(\partial M', \partial Q')$). Here M', Q' denote first-derived subdivisions.

Theorem. *For big k the $\mathbf{Z}/2$-prebundle $E(M, Q) \times D^k$ is locally trivial.*

Proof. By induction on $\dim M$, and starting with $\dim M = 0$, the proof follows that of 5.1 and 5.2 of [M-M]. □

To verify local triviality we consider the restriction of $E \times D^k$ to $D(v, M)$, v a vertex of M. The subcells of $D(v, M)$ are of the form $D(v\sigma, M)$, where $\sigma \in \mathrm{Link}(v, M)$, and the block of $E \times D^k$ lying over $D(v\sigma, M)$ is $D(v\sigma, Q) \times D^k$.

Unfortunately, $E|D(v, M)$ is not in the right form for us to be able to apply induction to it. However, via pseudo-radial projection of $\mathrm{Link}(v, Q')$ onto $\mathrm{Link}(v, Q)'$, $D(v\sigma, M)$ corresponds to $D(\sigma, M_0)$ and $D(v\sigma, Q)$ corresponds to $D(\sigma, Q_0)$ where M_0 and Q_0 are the simplicial complexes $\mathrm{Link}(v, M)$ and $\mathrm{Link}(v, Q)$. Since M is full in Q, M_0 is full in Q_0. This correspondence clearly respects involutions and, by means of it, the $\mathbf{Z}/2$-prebundle $E|D(v, M)$ corresponds to a $\mathbf{Z}/2$-prebundle (call it V) over the homology cell complex whose cells are the cells of M_0^* plus the additional cell $\mathrm{N}(v, M)$. In fact

$$V(D(\sigma, M_0)) = D(\sigma, Q_0) \quad \text{and} \quad V(\mathrm{N}(v, M)) = \mathrm{N}(v, Q).$$

Thus $V|M_0^* = E(M_0, Q_0)$. By induction on $\dim M$, the $\mathbf{Z}/2$-prebundle $V \times D^r|M_0^*$ is, for big r, a $\mathbf{Z}/2$-bundle.

We wish to show that V is stably trivial. So let a be any vertex of M_0 and write X for the complex consisting of all cells of M_0^* except $D(a, M_0)$. Since M_0 is a $\mathbf{Z}$-homology sphere one has $\widetilde{H}_*(X, \mathbf{Z}) = 0$ and therefore, as in the proof of Theorem 4.5 (Stable) above, $V \times D^k|X$ is trivial for big k.

To show V stably trivial it suffices to show that $\mathrm{Sph}(V)$ is stably trivial. Let J be an isomorphism between $\mathrm{Sph}(V \times D^k|X)$ and a product ε_X. We may glue $\mathrm{Sph}(V \times D^k)$ to J along $\mathrm{Sph}(V \times D^k|X)$ to obtain a $\mathbf{Z}$-homology manifold W, say. Then the boundary of $W \times [0, 1]$ contains $\mathrm{Sph}(V \times D^k) \times \{0\}$ and $\varepsilon_{X \times [0,1]}$ and these two are disjoint. Arguing now as in Proposition 5.1 of [M-M] we obtain an isomorphism between the $\mathbf{Z}/2$-prebundle $\mathrm{Sph}(V)$ and a product ε, as we wanted. In the terminology of [M-M], the above theorem can be expressed in the following concise and transparent way:

Theorem A. *The stable normal bundle of M in Q is equivariant.*

Definition. The stable isomorphism class of the $\mathbf{Z}/2$-bundle $E(M, Q) \times D^k$, for big k, constructed above is called the *stable normal* $\mathbf{Z}/2$*-bundle* of M in Q. For a given triangulation it clearly does not depend on k.

The *stable tangent* $\mathbf{Z}/2$*-bundle of* M is defined as follows. Triangulate M and let $Q = M \times M$ be subdivided equivariantly into simplexes without adding any new vertices, thus ensuring that the diagonal subcomplex is simplicially isomorphic to M. The stable tangent $\mathbf{Z}/2$-bundle is then defined to be the stable normal $\mathbf{Z}/2$-bundle of M in Q.

Note that, by construction, the restriction to ∂M of the stable normal $\mathbf{Z}/2$-bundle of M in Q is the stable normal $\mathbf{Z}/2$-bundle of ∂M in ∂Q.

It remains to establish uniqueness. By Theorem 4.12 (Stable) the stable normal $\mathbf{Z}/2$-bundle of M in Q corresponds to a unique stable isomorphism class of $\mathbf{Z}/2$ disk-bundles over the simplicial complex M' and hence, by Theorem 4.11, to a unique homotopy class of maps of M into $B\overline{H}$.

Theorem. *The homotopy class above depends only on the PL concordance class of the embedding of M in Q. In particular, it is independent of the particular triangulations of M and Q.*

Proof. As in [M-M, 5.4] with the obvious changes due to the involution.

3. Bordism of Z-homology manifolds

Before stating one main result (Theorem B) of this section we shall review some basic material on cobordism.

We begin by observing that a simple homology calculation shows that if $f: M \to K$ is a simplicial map from a $\mathbf{Z}$-homology manifold M to a simplicial complex K then $f^{-1}(x)$ is also a $\mathbf{Z}$-homology manifold, provided x lies in the interior of a top-dimensional simplex of K.

Singular bordism groups $\Omega_n^H(X)$ of a space X may be defined as usual for $\mathbf{Z}$-homology manifolds and, by the observation above, they give rise to a homology theory.

$\{M, f\}$, or simply $\{f\}$, will denote the bordism class of $f: M \to X$ in $\Omega_n^H(X)$ and $\{M\}$ the class of M in $\Omega_n^H = \Omega_n^H(\text{point})$. If M is a $\mathbf{Z}$-homology manifold (not necessarily orientable), M has a fundamental class $[M]$ in $H_n(M)$, where $H_*(X)$ means $H_*(X, \mathbf{Z}/2)$. If $\bar{t}_M: M \to B\overline{H}$ is a classifying map for the stable $\mathbf{Z}/2$ tangent bundle we write $\langle M\rangle = (\bar{t}_{M*}[M])$ in $H_n(B\overline{H})$. $\langle M\rangle$ does not depend on the choice of $\bar{t}_M$, and if $\{M\} = \{N\}$ then $\langle M\rangle = \langle N\rangle$.

Theorem B. *If M is a $\mathbf{Z}$-homology manifold then $\{M\} = 0 \Leftrightarrow \langle M\rangle = 0$.*

Proof. We write Ω_* instead of Ω_*^H. Let $h: \Omega_*(Y, B) \to H_*(Y, B)$ be the Hurewicz map, i.e., $h\{M \xrightarrow{f} Y\} = f_*[M]$. Let

$$t: \Omega_*(Y, B) \to \Omega_*(Y \times X, B \times X)$$

send $\{M, f\}$ to $\{M, (f, \bar{t}_M)\}$, where we write X for $B\overline{H}$. Finally, let P be the composite map

$$\Omega_*(Y, B) \xrightarrow{t} \Omega_*(Y \times X, B \times X) \xrightarrow{h} H_*(Y \times X, B \times X).$$

Theorem B may be rewritten thus:

Theorem. $P\colon \Omega_*(point) \to H_*(X)$ *is a monomorphism.*

We assume, by induction, the Theorem true for $* < n$, beginning trivially with $* = 0, 1$.

Lemma 1. *If* $P\{M\} = 0$ *then* $\{M, \bar{t}_M\} = \{N, c\}$ *in* $H_*(X)$, *where* c *is a constant map.*

Proof of Lemma 1. Choose a CW decomposition for X and write X^k for the k-skeleton. Given M we may assume (after a homotopy) that $\bar{t}_M(M) \subset X^n$. Let $0 < k \leq n$ and assume $t\{M\} = i_*\{f\}$, where $i\colon X^k \subset X$ is inclusion and $\{f\} \in \Omega_n(X^k)$. Let $C\{f\}$ be the image of $\{f\}$ in $\Omega_n(X^k, X^{k-1})$.

Lemma 1(a). *If* $C\{f\}$ *goes to zero in* $H_k(X, X^{k-1}) \otimes \Omega_{n-k}$ *in the diagram below, then* $t\{M\}$ *lies in* $\operatorname{Im} \Omega_n(X^{k-1})$.

Proof. Let Θ be the composite $(h \otimes \mathrm{id}) \circ \mu^{-1}$, where

$$\mu\colon \Omega_{k+1}(X^{k+1}, X^k) \otimes \Omega_{n-k} \to \Omega_{n+1}(X^{k+1}, X^k)$$

comes from the Ω_*-module structure of $\Omega_*(X^{k+1}, X^k)$. In the diagram

$$\begin{array}{ccc}
\Omega_{n+1}(X^{k+1}, X^k) & \xrightarrow[\cong]{\Theta} & H_{k+1}(X^{k+1}, X^k) \otimes \Omega_{n-k} \\
\downarrow \partial & & \\
\{f\}\Omega_n(X_k) & & \big\downarrow \partial\otimes\mathrm{id} \\
\downarrow \quad \downarrow & & \\
C\{f\}\Omega_n(X^k, X^{k-1}) & \xrightarrow[\cong]{\Theta} & H_k(X^k, X^{k-1}) \otimes \Omega_{n-k} \\
 & & \big\downarrow \\
 & & H_k(X^{k+1}, X^{k-1}) \otimes \Omega_{n-k} \\
 & & \big\downarrow \cong \\
 & & H_k(X, X^{k-1}) \otimes \Omega_{n-k}
\end{array},$$

the square commutes, by naturality of h and μ. If $\Theta C\{f\} \to 0$ in

$$H_k(X, X^{k-1}) \otimes \Omega_{n-k}$$

then $\Theta C\{f\}$ lies in the image of

$$H_{k+1}(X^{k+1}, X^k) \otimes \Omega_{n-k},$$

and hence $C\{f\}$ lies in the image of $\Omega_{n+1}(X^{k+1}, X^k)$. Thus $C\{f\} \to 0$ in $\Omega_n(X^{k+1}, X^{k-1})$, and so $\{f\}$ lies in the image of $\Omega_n(X^{k-1})$ as required. □

Proof of inductive step of Lemma 1. We first note that

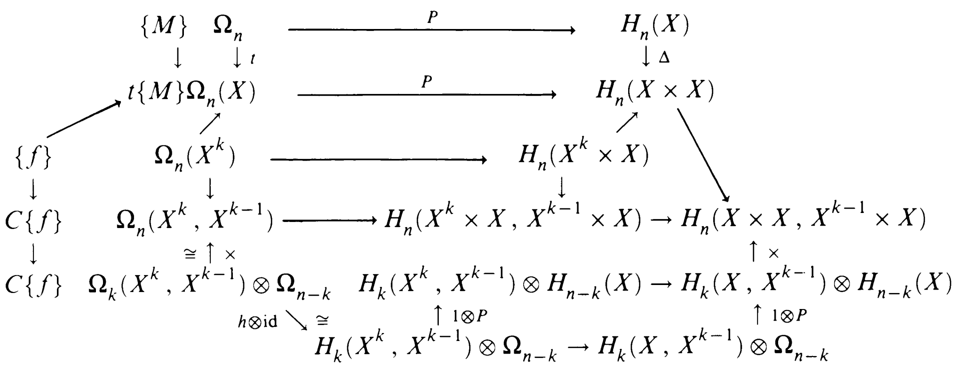
(i) $$H_k(X, X^{k-1}) \otimes \Omega_{n-k} \rightarrowtail H_k(X, X^{k-1}) \otimes H_{n-k}(X)$$

is a monomorphism, by induction.

(ii) $$H_k(X, X^{k-1}) \otimes H_{n-k}(X) \mapsto H_n(X \times X, X^{k-1} \times X)$$

is a monomorphism, by the relative Künneth theorem.

Thus, in the diagram below, we need only check that $C\{f\}$ goes to zero in $H_n(X \times X, X^{k-1} \times X)$:



Here $\times$ denotes external product and all unlabeled maps are induced in the obvious way. The diagram commutes. The only part which needs checking is the lower left-hand part. Take a representative $e \otimes \{V\}$ in $\Omega_k(X^k, X^{k-1}) \otimes \Omega_{n-k}$, and use the fact that

$$t\{e \times V\} = t\{e\} \times t\{V\} = \{\text{constant map}\} \times t\{V\}.$$

The proof of Lemma 1 is completed by following $C(f)$ around the diagram to deduce that $C(f)$ goes to zero in $H_n(X \times X, X^{k-1} \times X)$ as required. □

Let now $F: W \to X$ be the bordism between $\{\bar{t}_M\}$ and $\{N, c\}$ given by Lemma 1. Since W is compact one has $F(W) \subset B\overline{H}(a+1)$ for large a. Then the $\mathbf{Z}/2$ disk-bundle $\xi = F^*(\gamma_a)$ over W is such that $\xi|M \cong \bar{\tau}_M$ and $\xi|N$ is trivial (as $\mathbf{Z}/2$-bundles). Here γ_a is the universal bundle over $B\overline{H}(a+1)$ and $\bar{\tau}_M$ is the stable $\mathbf{Z}/2$ tangent bundle of M with fibre $\mathbf{D}^{a+1}$, which we know to be an equivariant tubular neighbourhood of $\Delta M \times \{0\}$ in $P = M \times M \times \mathbf{D}^{a-n+1}$. Excise the interior of (the total space of) $\bar{\tau}_M$ from P and attach $\mathrm{Sph}(\xi)$ along $\mathrm{Sph}(\xi|M)$ via the $\mathbf{Z}/2$-isomorphism $\mathrm{Sph}(\xi|M) \cong \mathrm{Sph}(\bar{\tau}_M)$. If $\widehat{P}$ is the resulting homology manifold, then there is, by construction, a free involution on $\widehat{P}$. Furthermore, we have

$$\partial\widehat{P} = \partial P \amalg \mathrm{Sph}(\xi|N) = (M \times M \times S^{a-n}) \amalg \mathrm{Sph}(\xi|N).$$

Since $\xi|N$ is trivial, there is a bordism with a free involution between $\mathrm{Sph}(\xi|N)$ and $N \times S^a$. Glueing this bordism to $\widehat{P}$ and quotienting out by the involution (called Θ from now on), we obtain a homology manifold Q with boundary ∂Q given by the disjoint union of $N \times \mathbf{P}^a$ and $R = (M \times M \times S^{a-n})/\Theta$ ($\mathbf{P}$ stands for projective space). Let $f: Q \to \mathbf{P}^b$ (large b) be a classifying map for the involution Θ. Since the projections $M \times M \times S^{a-n} \to S^{a-n}$ and $N \times S^a \to S^a$ are Θ-equivariant, and $n > 0$, we may assume, up to homotopy, that $f(R) \subseteq \mathbf{P}^{a-1} \subset \mathbf{P}^b$ and $f|N \times \mathbf{P}^a$ is a projection onto $\mathbf{P}^a \subset \mathbf{P}^b$. Now take a complementary projective subspace P_1^{b-a} meeting $\mathbf{P}^a$ transversely in a point and such that $\mathbf{P}_1 \cap P^{a-1} = \phi$, and note that $f|\partial Q$ is transverse to $\mathbf{P}_1$ with inverse image N (see remark below on transversality). Then make f transverse to $\mathbf{P}_1$ modulo ∂Q so as to have a homology manifold $f^{-1}(\mathbf{P}_1)$ with boundary $f^{-1}(\mathbf{P}_1 \cap \mathbf{P}^a) = N$. Thus N, and therefore our original M, bounds a homology manifold. This completes the proof of Theorem B. □

Remark. Transversality for maps of homology manifolds can be defined and holds when the ambient space is a (genuine) PL manifold. This is observed explicitly, for instance in [Quinn], and can be made to follow from Stone's PL stratification theory or, more simply, from the general treatment of transversality given in [B-R-S, II §4].

The splitting $S: BH \to B\overline{H}$.

Definition. Two S^{n-1}-bundles E/K and F/L where $|K| = |L|$ are said to be *equivalent* (written $E \sim F$) if they define the same homotopy class $|K| \to BH(n)$.

Note that E and F are not assumed to be equivariant.

Theorem 1. *Let $P \subset M \subset Q$ be proper sub(homology) manifolds, E the normal bundle of P in M, F the normal bundle of M in Q, and G that of P in Q. Then*

$$E \oplus (F|P) \sim G.$$

This is the homology-manifolds version of Corollary 4.9 of [R-S] (its proof is, however, different).

Proof. Let K be the base space of E and let F be the normal bundle over the dual cell-complex coming from a triangulation J of M in Q, where J contains a subcomplex J_1, with $|J_1| = E$, and such that the cells of K and the blocks of E are subcomplexes. Let H be a bundle over $J^* \times I$ such that $H|J^* \times \{0\} = F$ and $H|J^* \times \{1\}$ is the amalgamation of a bundle F' over J. Consider the bundle F'' over K, constructed as follows: for each cell C of K, $F''(C)$ is the union of those blocks of F' which lie over $E(C)$ (since $E(C)$ is contractible, F'' is a bundle). In the terminology of [Maunder, 2.3] we have

$$F'' = E \circ F' \qquad \text{(composition)}.$$

Using [Maunder, 2.5] one has

$$F'' \sim E \oplus (F'|K) \sim E \oplus (F|K).$$

Furthermore, the tubular neighborhood theorem [M-M, 5.5] implies that F'' is a normal bundle of P in the (homology) manifold F' and hence equivalent to the normal bundle of P in Q, the equivalence being given by taking the normal bundle of $P \times I$ in the manifold H above. □

Let E/K be a bundle, u its equivalence class, $|K| = M$, $\mathscr{T}$ the (nonequivariant) tangent bundle.

Theorem 2. $\mathscr{T}_E|M = u \oplus \mathscr{T}_M$. *This is the analogue of Theorem* 5.5 *of* [R-S], *whose proof may be adapted to our case* (*by using Theorem* 1) *once we note that* u *has a stable inverse given by the* H-*space structure of* BH.

We now dispose of all the ingredients necessary to argue as in [Mann-Miller, 2.3–2.5] and to obtain analogous results for homology bundles.

Theorem 3. *Stably every homology bundle admits an involution in a natural way, i.e., the "forgetful map"* $F\colon B\overline{H} \to BH$ *has a section* $S\colon BH \to B\overline{H}$.

Proof. As in [Mann-Miller, 2.3] using Theorem 1 above. □

Definition. Let E be a $\mathbf{Z}/2$-bundle over X. E is *stably standard* if the classifying map $f\colon X \to B\overline{H}$ factors

$$\begin{array}{ccc} & \overset{f}{\nearrow} & B\overline{H} \\ X & & \uparrow S \\ & \underset{f_1}{\searrow} & BH \end{array}$$

where f_1 is the classifying map for E as a bundle without involution.

Corollary (Compare [Mann-Miller, 2.5]). *If* M *is a* $\mathbf{Z}$-*homology manifold and* $\overline{\mathscr{T}}_M$ *its stable* $\mathbf{Z}/2$ *tangent bundle, then the natural involution of* $\overline{\mathscr{T}}_M$ *given by Theorem* 3 *is stably standard.*

Let now $t_M\colon M \to BH$ be a classifying map for the stable (nonequivariant) tangent bundle of M. The following theorem is the main result of this work.

Theorem C. $\{M\} = 0$ *if and only if* $(t_M)_*[M] = 0$ *in* $H_n(BH)$.

Proof. Since $\overline{\mathscr{T}}_M$ is stably standard we may assume that the composite $M \xrightarrow{t_M} BH \xrightarrow{S} B\overline{H}$ classifies the stable $\mathbf{Z}/2$ tangent bundle of M. Since S is a section of $B\overline{H} \to BH$, S_* is a monomorphism and the theorem follows from Theorem B. □

4. Extension to homology manifolds with a fixed system of singularities

We assume the treatment of singularities given in [B-R-S, IV, §3]. Let $\mathscr{L}_H$ be a multiplicative class of links such that each $|L| \in \mathscr{L}_H$ is also a homology sphere. In analogy with the case of PL manifolds an element $|L| \in \mathscr{L}_H$ will be called an $\mathscr{L}_H$-*sphere*, and a cone $|aL|$ an $\mathscr{L}_H$-*disk*.

Definition. An $\mathscr{L}_H$*-cell* C is a simplicial complex of the type aL, where L is a triangulation of an $\mathscr{L}_H$*-sphere* or of an $\mathscr{L}_H$-disk. The *boundary* ∂C is $L \cup (a\partial L)$ (only L if $|L|$ is an $\mathscr{L}_H$-sphere).

We note that $|C| \cong |x\partial C|$ for some $x \in |\overset{\circ}{C}|$ and, furthermore, $|\partial C| \in \mathscr{L}_H$. Because let $C = aL$. If $|L| \in \mathscr{L}_H$ then $|\partial C| = |L| \in \mathscr{L}_H$, while if $|L|$ is an $\mathscr{L}_H$-disk $|aL'|$, then $|\partial C| \cong S^0 * |L'|$ and therefore lies in $\mathscr{L}_H$. As a consequence, we have that an $\mathscr{L}_H$-cell C is an $\mathscr{L}_H$-manifold with boundary ∂C.

A *complex* of $\mathscr{L}_H$-cells (briefly, an $\mathscr{L}_H$*-complex*) is a homology cell complex in which each cell is an $\mathscr{L}_H$-cell. The main example is the complex of dual cones in a triangulation of an $\mathscr{L}_H$-manifold, other examples are simplicial complexes and products $K \times L$ of $\mathscr{L}_H$-complexes.

A *locally trivial* $\mathscr{L}_H$*-prebundle* over an $\mathscr{L}_H$-complex is a homology bundle in which blocks and trivializations are $\mathscr{L}_H$-manifolds.

The correspondence which to an $\mathscr{L}_H$-bundle E associates its sphere bundle $\mathrm{Sph}(E)$ does not seem to induce a bijection between isomorphism classes, because, in general, the coning procedure used in the second part of the proof of [M-M, 3.3] does not extend to $\mathscr{L}_H$-manifolds. However, this causes no real problem to us, because one can work with sphere bundles throughout.

The whole theory of homology bundles developed in [M-M] continues to work for $\mathscr{L}_H$-bundles, as the reader can convince himself by a careful inspection of the proofs. There are only a couple of important points that are worth mentioning, the first being

Proposition 3.4 [M-M] ($\mathscr{L}_H$-version). *Let E be an $\mathscr{L}_H$-bundle with fibre S^n over an $\mathscr{L}_H$-complex K with $|K|$ an $\mathscr{L}_H$-manifold of* $\dim m$. *Then E is an $\mathscr{L}_H$-manifold of* $\dim(m+n)$ *with $\partial E = E(\partial K)$.*

Proof. Using "transverse stars" as in [B-R-S, II 1, 2], one has {neighborhood of x in E} $\cong$ {neighborhood of x in $E(C)$} $\times$ {upper transverse star} $\cong$ {neighborhood of x in $E(C)$} $\times$ {lower transverse star} $\cong$ {neighborhood of x in $E(C)$} $\times$ $\{D(\sigma, K)\}$, where $x \in \operatorname{Int} E(C)$ for some $C \in K$ and σ is a top dimensional simplex of C. It follows that

$$\mathrm{Link}(x, E) \cong \mathrm{Link}(x, E(C)) * \dot{D}(\sigma, k),$$

and the latter polyhedron is in $\mathscr{L}_H$ by hypothesis and multiplicativity. The result is proved.

The other point to check is the $\mathscr{L}_H$-version of Theorem 3.5 of [M-M], which is stated and proved in the same way as in [M-M] once we have noted the following (notation as in [M-M]):

$$Y = X \times \{1\} \cup \partial D \times I \cup \partial D \times I \cup D$$

is PL isomorphic to $|\partial C|$ and therefore $Y \in \mathscr{L}_H$. Furthermore Z is isomorphic to the "double" of C, namely $|C| \cup_\partial |C|$, which in turn is PL isomorphic to $S^0 * |\partial C|$ and thus it lies in $\mathscr{L}_H$ because $|\partial C|$ does. One has a classifying Δ-set $B\mathscr{L}_H(n)$ for the Δ-semigroup $\mathscr{L}_H(n)$, whose k-simplexes are the $\mathscr{L}_H$-automorphisms of the product $\Delta^k \times S^{n-1}$, and a bijection $(A\mathscr{L}_H)_n(K) \cong [|K|, B\mathscr{L}_H(n)]$, where K is any $\mathscr{L}_H$-complex and $(A\mathscr{L}_H)_n(-)$ denotes the set of isomorphism classes of $\mathscr{L}_H$-bundles with fibre S^{n-1}.

The whole theory of normal bundles developed in [M-M] is still valid in the case of $\mathscr{L}_H$-singularities, and one deduces that the homology normal bundle of an $\mathscr{L}_H$-manifold M properly embedded in an $\mathscr{L}_H$-manifold Q is, in fact, an $\mathscr{L}_H$-bundle, and the homotopy class of its classifying map into $B\mathscr{L}_H$ depends only on the PL concordance class of the embedding $M \subset Q$ (in particular the homology tangent bundle of an $\mathscr{L}_H$-manifold is an $\mathscr{L}_H$-bundle). Even an $\mathscr{L}_H$-version of the Tubular Neighborhood Theorem 5.5 of [M-M] holds.

Finally, it is clear how to introduce involutions in the context of $\mathscr{L}_H$-bundles, which, through the methods described in the previous sections, leads to the following more precise version of Theorem C.

Theorem C ($\mathscr{L}_H$). *A closed $\mathscr{L}_H$-manifold M^n bounds an $\mathscr{L}_H$-manifold if and only if $(\tau_M)*[M] = 0$ in $H_n(B\mathscr{L}_H)$ (here the notation has an obvious meaning).*

As the last thing, we should make sure that the above theorem gives back the well-known result of Browder et al. on PL characteristic numbers of PL manifolds, once we take $\mathscr{L}_H$ to be the class of PL spheres. But in this case $B\mathscr{L}_H(n)$ is, by definition, the classifying space of the Δ-semigroup PL_H, of which a typical k-simplex is a block preserving H-cobordism by PL manifolds between $\Delta^k \times S^{n-1}$ and itself. Therefore it is enough to establish

Proposition. *BPL_H is homotopy equivalent to BPL.*

Proof. There is a homotopy fibration $PL_H/PL \to BPL \to BPL_H$, and

$$\pi_k(PL_H/PL) = 0$$

by [Martin, Lemma 1] with $n \gg k$.

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References

G. A. Anderson, *Surgery on Λ-homology manifolds*, Illinois J. Math. **24** (1980), 653–675.

[B-R-S] S. Buoncristiano, C. Rourke and B. J. Pand Sanderson, *A geometric approach to homology theory*, LMS Lecture Notes Ser. 18, London Math. Soc., 1976.

[B-H] S. Buoncristiano and D. Hacon, *Elementary proofs of two theorems of Thom*, Topology **20** (1981).

——, *A geometrical approach to characteristic numbers*, Geometry Today, Birkhäuser, 1985, pp. 21–38.

N. Martin, *On the difference between homology and piecewise-linear bundles*, J. London Math. Soc. **6** (1973), 197–204.

[M-M] N. Martin and C. R. F. Maunder, *Homology cobordism bundles*, Topology **10** (1971), 93–110.

B. M. Mann and E. I. Miller, *On mapping tori and projective bundle constructions in PL cobordism*, Indiana Univ. Math. J. **31** (1982), 225–271.

C. R. F. Maunder, *On the Pontryagin classes of homology manifolds*, Topology **10** (1971), 111–118.

F. Quinn, *Semifree group actions and surgery on PL homology manifolds*, Geometric Topology, Lecture Notes in Math., vol. 438, Springer-Verlag, New York, 1975.

[R-S] C. P. Rourke and B. J. Sanderson, *Block bundles* II, Ann. of Math. **87** (1968), 256–278.

M. M. Cohen, *Simplicial structures and transverse cellularity*, Ann. of Math. **85** (1967).

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